On Notion of Foundations of Navigation in Maritime Education

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Abstract The problem of travelling, ordered motion and related transportation goes back to long-ago times. It has been appearing in changing forms since then. Its aspects were called in different ways during the centuries in the past. Together with the following progress the nature of navigation was also becoming more Abstract. How is it nowadays? Looking at the navigation problem submitted by Ernst Zermelo (1871 - 1953) we propose to generalize it. Analysis of the evaluation of the long standing problem goes through historical and present concepts based on the practical applications as well as the modern theories. Conclusions concern the view at the present trends and obtained results forming the ideas of nearest future with reference to maritime education.

Keywords navigation; maritime education; Zermelo; geometry;

0 Introduction

It is hard to state exactly when people started to travel and how they guided themselves to reach chosen destination. Let us say that as far as remembered the human being had been travelling. Of course the meaning of "travelling" can be different. It depends on the point of view at the problem. When we focus on the different scientific disciplines we find out that the problem of travelling is present and evergreen there. One of its aspects is navigation. There are several kinds of navigation which are studied nowadays considering the applications. The progress made in transportation caused the different types of navigation appeared. We try to focus on maritime part however it is worth of remembering it is a part of navigation in general. Thus, can we say if there is one navigation as research discipline nowadays?

It is necessary to point out that the time of writing this article influenced considerably on its contents. Analyzing the same subject for instance in the 19th century it would not have included other types of navigation which appeared since then. Then the maritime part was the most significant and the created models treated on its applications. However we try to look at the concerned subject in general way. So first things first.

1 Programme of navigational lectures at maritime university

The studies at maritime university take usually four or five years including approximately twelve months of sea practice. The programme of lectures of navigation which is obviously the most important at faculties of navigation includes e.g.

- terrestrial navigation with the analytical and geometrical aspects;
- great circle and rhumb line navigation;
- accuracy in navigation;
- astronomical navigation;
- tides and tidal streams calculations;
- Electronic Chart Systems, Electronic Chart Display and Information System and other electronic aids to navigation (including e.g. satellite systems);
- voyage planning (including navigation in constrained waters or ice) and optimization;
- automation of calculus (different coordinate systems) and computer systems in navigation.

All the lectures follow the standards according to respective conventions as STCW or SOLAS. Additionally there is also a background of the compulsory lectures on mathematics, physics, cartography, geodesy, electronics, computer sciences, meteorology or oceanography. The programme has to be flexible. That means ready to implement the new solutions prepared by international bodies. From time to time the programme needs changes and all the time should be evaluated.

There are of course other groups of lectures obligatory for students to know the job on board the ship properly. However it is not our goal to list all other important disciplines which are necessary for maritime purposes but to focus on navigation itself. The contents of the programmes all over the world are very similar according to international criteria with accepted differences in their time organization. And this is all referring to applications basing on newer and newer solutions implemented in the maritime industry. The students need the part of knowledge compulsory to follow the above mentioned lectures. We propose another one which should not be avoided. This can be found in such research disciplines as mathematics and physics. However is there a space in education programmes for the general view at navigation we are looking for below?

2 Historical background of navigational problems

Research on 5th Euclid’s postulate (parallel postulate) caused discovery of non-Euclidean geometries. Given any straight line and a point not on it, there exists one and only one straight line which passes through that point and never intersects the first line, no matter how far they are extended. This statement is equivalent to the fifth of Euclid's postulates, which Euclid himself avoided using until proposition 29 in his *Elements*. For centuries, many researchers believed that this statement was not a true postulate, but rather a theorem which could be derived from the first four of Euclid's postulates. That part of geometry which could be derived using only postulates 1-4 came to be known as absolute geometry. Over the years, many purported proofs of the parallel postulate were published. However, none were correct, including the 28 "proofs" G. S. Klügel analyzed in his dissertation of 1763. The main motivation for all of this effort was that Euclid's parallel postulate did not seem as intuitive as the other axioms, but it was needed to prove important results. John Wallis proposed a new axiom that implied the parallel postulate and was also intuitively appealing. His axiom states that any triangle can be made bigger or smaller without distorting its proportions or angles. Wallis's axiom never caught on. In 1823, Bolyai and Lobachevsky independently realized that entirely self-consistent non-Euclidean geometries could be created in which the parallel postulate did not hold. Gauss had also discovered but suppressed the existence of non-Euclidean geometries. As stated above, the parallel postulate describes the type of geometry now known as parabolic geometry. If, however, the phrase "exists one and only one straight line which passes" is replaced by "exists no line which passes," or "exist at least two lines which pass," the postulate describes equally valid (though less intuitive) types of geometries known as elliptic and hyperbolic geometries, respectively. The parallel postulate is equivalent to the equidistance postulate, Playfair's axiom, Proclus' axiom, the triangle postulate, and the Pythagorean theorem. There is also a single parallel axiom in Hilbert's axioms which is equivalent to Euclid's parallel postulate. S. Brodie has shown that the parallel postulate is equivalent to the Pythagorean theorem.

Problem of finding the shortest route has already been considered by the ancients for practical reasons. Looking back in the past it can be said that the idea came back many times but in different forms accordingly to appearance of new problems or new circumstances. The roots of calculus of variations go back to Fermat's principle of least time (1662). However, the problem which gave the development of the field its first momentum was the celebrated brachystochrone problem stated by Galileo (1638) and solved by John Bernoulli. Bernoulli's paper on the problem was published in the Acta Eroditorum Lipsae (1696)[10]. The brachystochrone problem can be stated as follows:

Consider a curve in a vertical plane, starting at a given point A and ending at another given point B, located at a lower ordinate. Point mass slides along such a curve, under the effect of gravity, so moving from A to B in some interval of time (*transfer time*)*. Which is the curve connecting A to B resulting in the shortest transfer time?*

Following the successful solution of the brachystochrone problem, many other variational problems were studied in the subsequent centuries, from the search of geodesic lines to isoperimetric problems, from nautical paths in stationary sea currents to the Zermelo navigation problem[6].

On a cold New Year Eve of 1720, Count Jacopo Francesco Riccati, a nobleman living in the Republic of Venice, wrote a letter to his friend Giovanni Rizzetti, where he proposed two new differential equations. In modern symbols, these equations can be written as follows:

$$\dot{x} = \alpha x^2 + \beta t^m \tag{1}$$

$$\dot{x} = \alpha x^2 + \beta t + \gamma t^2 \tag{2}$$

where *m* is a constant and *t* is the independent variable. This is probably the first document witnessing the early days of the Riccati equation which was to become of paramount importance in our days. Riccati's main interest in the area of differential equations focused on the methods of separation of variables. Such an interest originated in the reading of Gabriele Manfredi's book *De constructione aequationum differentialium primi gradus* printed in Bologna in 1707. Originally, Riccati attention focused on the following problem of geometric type: *suppose that a point of coordinates* (α ,β) *describes a trajectory in the plane according to a linear differential equation of the first order*. Once Riccati days had passed, his equation has been studied by many, in particular Euler (ca. 1760) and Jacques Liouville (ca.1840). However, it is in the 20th century that the equation reaches a paramount importance, especially for the developments of calculus of variations and optimal filtering and control. While the literature on the second variation methods in calculus of variations mainly developed in the first half of the 20th century, optimal filtering and control entered the scientific stage with Kalman's contribution mainly during the decade 1960-1970, and stimulated the research activity for the remaining portion of the century. The problem dealt with in this new field is also a functional optimization problem, but, a new challenging issue is there; the presence of exogenous variables, affecting the dynamics of a phenomenon described in state-space form.

Under the impetus of Kalman work, such a modelization of the real world had to become of paramount importance for diverse fields of investigation and engineering application for the years to come. Thanks to such a unifying model, phenomena, plants, devices, processes did not need any more a conceptual diversification. Another characteristic of Kalman modelization is the possibility of easily incorporating the effect of disturbances, a major ingredient for the formidable communication and information problems of the 20th century. Finally, we should note that, while the independent variable in calculus of variations is typically thought of as a space coordinate, the typical independent variable of optimal filtering and control is time. Under state-space representations, the optimality conditions are naturally posed in terms of Riccati equations[6].

Hilbert's list of problems, read at the International Congress of Mathematicians in 1900, is perhaps one of the most influential documents in the history of mathematics. The twenty-three problems in this list have been the subject of numerous investigations for the last hundred years and continue to yield much beautiful mathematics. Even when one of Hilbert's problems has been solved in its original formulation, its variations and the developments arising from its solution continue to pique the curiosity of mathematicians. One example is Hilbert's third problem on the decomposition of polyhedra. This problem was solved by M.Dehn just two years after Hilbert's address, but the concepts he introduced have evolved in different directions. For instance, the theory of valuations, a central part of modern convex geometry, is a direct descendent of Dehn's solution. In posing his problems, Hilbert did not shy away from vague statements which would be subject to interpretation. Problem six on the mathematical treatment of the axioms of physics is perhaps the prime example of this. Hilbert's fourth problem is another. In its original formulation, Hilbert's fourth problem asks *to construct and study the geometries in which the straight line segment is the shortest connection between two points*. The original wording makes one think the problem is part of Hilbert's project to study the foundations of geometry. However, the different modern approaches make it clear that the problem is at the basis of integral geometry, inverse problems in the calculus of variations, and Finsler geometry[14].

Probably we could add more significant examples of the past problems concerning field of navigation however they cannot be reported here due to lack of space. It is worth of reminding that one of the first Leonhard Euler's papers was focused on marine navigation. We can see the foundations of navigation refer to geometries which differ. The fact has been well known since 19th century. We conclude mentioning that the scientific research changes the researched object.

3 Mathematical models in navigation and its applications

Besides their pedagogical value, many map projections are invaluable for specialized professionals. A common problem is finding the shortest route across the Earth surface between two points. Such path is always part of a geodesic or *great circle* on the globe surface. The geodesic is used by ship and aircraft navigators attempting to minimize distances, while radio operators with directional antennae used to look for a bearing yielding the strongest signal. For many purposes, it is entirely adequate to model the earth as a sphere. Actually, it is more nearly an oblate ellipsoid of revolution, also called an oblate spheroid. This is an ellipse rotated about its shorter axis. Compared to a sphere, an oblate spheroid is flattened at the poles. The earth's flattening is quite small, about 1 part in 300, and navigation errors induced by assuming the earth is spherical, for the most part, do not exceed this, and so for many purposes a spherical approximation may be entirely adequate. On a sphere, the commonly used coordinates are latitude and longitude, likewise on a spheroid, however on a spheroid one has to be more careful about what exactly one means by latitude[1].

The great Flemish cartographer Gerhard Kremer became famous with the Latinized name Gerardus Mercator. A revolutionary invention, the cylindrical projection bearing his name has a remarkable property: any straight line between two points is a loxodrome, or line of constant course on the sphere. In the common equatorial aspect, the Mercator loxodrome bears the same angle from all meridians. In other words, if one draws a straight line connecting a journey's starting and ending points on a Mercator map, that line's slope yields the journey direction, and keeping a constant bearing is enough to get to one's destination. The only conformal cylindrical projection, Mercator's device was a boon to navigators from the 16th-century until the present, despite suffering from extreme distortion near the poles: Antarctica is enormously stretched, and Greenland is rendered about nine times larger than actual size. Indeed, stretching grows steadily towards the top and bottom of the map (in the equatorial form, in higher latitudes; the poles would be actually placed infinitely far away). Although important, a Mercator map is not the only one used by navigators, as the loxodrome (rhumb line) does not usually coincides with the geodesic. This projection was possibly first used by Etzlaub (ca. 1511). However, it was for sure only widely known after Mercator's atlas of 1569. Mercator probably defined the graticule by geometric construction. E.Wright formally presented equations in 1599. More commonly applied to large-scale maps, the transverse aspect preserves every property of Mercator's projection, but since meridians are not straight lines, it is better suited for topography than navigation. Equatorial, transverse and oblique maps offer the same distortion pattern. The transverse aspect, with equations for the spherical case, was presented by Lambert in his seminal paper (1772). The ellipsoidal case was developed, among others, by the great mathematician Carl Gauss (ca. 1822) and Louis Krüger ca. 1912. It is frequently called the Gauss conformal or Gauss-Krüger projection[16].

The vessel or aircraft can reach its destination following the fixed bearing along the whole trip disregarding some obvious factors like for instance weather, fuel range, geographical obstructions. However, that easy route would not be the most economical choice in terms of distance. The two paths almost coincide only in brief routes. Although the rhumb line is much shorter on the Mercator map, an azimuthal equidistant map tells a different story, even though the geodesic does not map to a straight line since it does not intercept the projection centre. Since there is a trade-off: following the geodesic would imply constant changes of direction (those are changes from the current compass bearing and are only apparent, of course: on the sphere, the trajectory is as straight as it can be). Following the rhumb line would waste time and fuel. So a navigator could follow a hybrid procedure[15]:

- trace the geodesic on an azimuthal equidistant or gnomonic map;
- break the geodesic in segments;
- plot each segment onto a Mercator map;
- use a protractor and read the bearings for each segment;
- navigate each segment separately following its corresponding constant bearing.

The shortest path between two points on a smooth surface is called a geodesic curve on the surface. On a flat surface the geodesics are the straight lines, on a sphere they are the great circles. Remarkably the path taken by a particle sliding without friction on a surface will always be a geodesic. This is because a defining characteristic of a geodesic is that at each point on its path, the local center of curvature always lies in the direction of the surface normal, i.e. in the direction of any constrained force required to keep the particle on the surface. There are thus no forces in the local tangent plane of the surface to deflect the particle from its geodesic path. There is a general procedure, using the calculus of variations, to find the equation for geodesics given the metric of the surface[1].

The Earth is not an exact ellipsoid, and deviations from this shape are continually evaluated. The geoid is the name given to the shape that the Earth would assume if it were all measured at mean sea level. This is an undulating surface that varies not more than about a hundred meters above or below a well-fitting ellipsoid, a variation far less than the ellipsoid varies from the sphere. The choice of the reference ellipsoid used for various regions of the Earth has been influenced by the local geoid, but large-scale map projections are designed to fit the reference ellipsoid, not the geoid. The selection of constants defining the shape of the reference ellipsoid has been a major concern of geodesists since the early 18th century. Two geometric constants are sufficient to define the ellipsoid itself e.g. the semimajor axis and the eccentricity. In addition, recent satellite-measured reference ellipsoids are defined by the semimajor axis, geocentric gravitational constant, and dynamical form factor, which may be converted to flattening with formulas from physics.

In the early 18th century, Isaac Newton and others concluded that the Earth should be slightly flattened at the poles, but the French believed the Earth to be egg-shaped as the result of meridian measurements within France. To settle the matter, the French Academy of Sciences, beginning in 1735, sent expeditions to Peru and Lapland to measure meridians at widely separated latitudes. This established the validity of Newton's conclusions and led to numerous meridian measurements in various locations, especially during the 19th and 20th centuries. Between 1799 and 1951 there were 26 determinations of dimensions of the Earth.

There are over a dozen other principal ellipsoids, however, which are still used by one or more countries. The different dimensions do not only result from varying accuracy in the geodetic measurements (the measurements of locations on the Earth), but the curvature of the Earth's surface (geoid) is not uniform due to irregularities in the gravity field.

Until recently, ellipsoids were only fitted to the Earth's shape over a particular country or continent. The polar axis of the reference ellipsoid for such a region, therefore, normally does not coincide with the axis of the actual Earth, although it is assumed to be parallel. The same applies to the two equatorial planes. The discrepancy between centers is usually a few hundred meters at most. Satellite-determined coordinate systems are considered geocentric. Ellipsoids for the latter systems represent the entire Earth more accurately than ellipsoids determined from ground measurements, but they do not generally give the best fit for a particular region. The reference ellipsoids used prior to those determined by satellite are related to an initial point of reference on the surface to produce a datum, the name given to a smooth mathematical surface that closely fits the mean sea-level surface throughout the area of interest. The initial point is assigned a latitude, longitude, elevation above the ellipsoid, and azimuth to some point. Satellite data have provided geodesists with new measurements to define the best Earth-fitting ellipsoid and for relating existing coordinate systems to the Earth's center of mass. For the mapping of other planets and natural satellites, Mars is treated as an ellipsoid. Other bodies are taken as spheres, although some irregular satellites have been treated as triaxial ellipsoids and are mapped orthographically[15].

The metric has already been mentioned without its definition. It is important to remember that obtained shortest distance (geodesics) also depends on the type of metric we use on the considered surface in navigation. A function $g: X \times X \rightarrow R_+ = \{x \in R \mid x \geq 0\}$ is called a metric (or distance) in X, if (Viro, Ivanov, Netsvetaev, Kharlamaov[17])

(1) $g(x,y)=0$, if and only if $x = y$ (positivity);

(2) $g(x,y) = g(y,x)$ for every $x, y \in X$ (symmetry);

(3) $g(x,y) \leq g(x,z) + g(z,y)$ for every $x, y, z \in X$ (triangle inequality).

Metric is a nonnegative function describing the "distance" between neighboring points for a given set. The pair (X, g), where g is a metric in set X, is called a metric space. The geodesics can look different even on the plane if different metrics are taken into consideration what is presented in Fig.1. It is easy to see that mentioned brachystochrone problem belongs to the group of problems characteristic for the field of navigation.

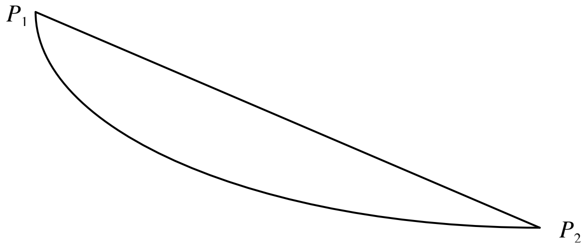

Fig. 1 Geodesics on the plane in different metrics

If we consider the Euclidean metric on the plane, the shortest distance is presented as a segment of a straight line P_1P_2 in the Fig.1. However the arc P_1P_2 can be geodesic when considering a different metric. In the pure geometrical research we don't consider time. To consider time means to consider the speed (velocity) vector. Actually the brachystochrone problem the criterion was time. The curve of the shortest time occurred to be the part of cycloid. Roughly speaking a geodesic is a curve whose length is the shortest distance between two points as already mentioned. This notion makes sense not only for surfaces in R^3 but also for Abstract surfaces and more generally (Riemannian) manifolds. The distance between two points is the length of the path connecting them. In general, the distance between points x and y in a Euclidean space R^n is given by

$$d = |x - y| = \sqrt{\sum_{i-1}^{n} |x_i - y_i|^2} \tag{3}$$

For curved or more complicated surfaces, the metric can be used to compute the distance between two points by integration. The distance generally means the shortest distance between two points. For example, there are an infinite number of paths between two points on a sphere but, in general, only a single shortest path. The spherical distance between two points A and B of a sphere is the distance of the shortest path along the surface of the sphere (paths that cut through the interior of the sphere are not allowed) from A to B, which always lies along a great circle. Most often the research and calculus in navigational literature are considered on the spherical or spheroidal models of Earth because of practical reasons. The flow of geodesics on the ellipsoid (spheroid) differs from the geodesics on the sphere. There are known different geodesics on the same surface, with the same metric considered. However geodesic refers to the metric what is usually not taken into consideration in the navigational lectures. And there are different flows of geodesics on the same surface when different metrics are considered. That means we can obtain very interesting results in navigational aspect if we change the researched object with its geometrical and physical features.

The mathematical formulas used in navigation and studied at the maritime universities are known well. However if we considered different shape of the planet (surface) the formulas could be quiet different. Let us imagine that the vessels do not sail on spherical (or spheroidal) earth but torus – shaped planet. In this case the flow of geodesics and mentioned rhumb line or used charts are based on other mathematical expressions (different geometrical object). The torus is topologically more simple than the sphere, yet geometrically it is a very complicated manifold indeed. The round torus metric is most easily constructed via its embedding in a Euclidean space of one higher dimension.

4 Zermelo's problem of navigation

Ernst Zermelo was a legend already during his lifetime. Today and also at that time his name was connected to the big debate on the axiom of choice used by him in 1904 for the proof of the well-ordering theorem. He was responsible for the axiomatization of set theory presented in 1908, which helped to establish set theory as a widely accepted mathematical theory. Today Zermelo's name is omnipresent in set theory in acronyms like "ZF" (Zermelo – Fraenkel system) or "ZFC" (Zermelo – Fraenkel [axiom of] Choice). But Zermelo was more than the founder of axiomatized set theory. With his application of set theory to the theory of chess, he became one of the founders of game theory, and with his application of the calculus of variations to the problem of the navigation of aircrafts he pioneered navigation theory. With his recurrence objection in kinetic gas theory he annoyed Ludwig Boltzmann, and with a translation of Homer's Odyssee he pleased even philology experts. In short: Zermelo was famous in his time [18].

Navigation problem was posed by Zermelo in 1931. Let a vessel travelling at constant speed navigate on a body of water having surface velocity. The navigation problem asks for the course which travels between two points in minimal time. Zermelo assumed that the open sea was R^2 with the Euclidean metric. This problem is thought to be a time optimal control problem which consists of finding the quickest nautical path of a ship in the presence of stationary sea currents or wind. The sea surface can be modeled by a two-dimensional Riemannian surface *M* and the currents by an autonomous vector field $X \in Vec\ M$. Dynamics of optimal trajectories for Zermelo problem are given by

$$\dot{q} = X(q) + \cos u e_1 + \sin u e_2, \quad q \in M,\ u \in S^1 \tag{4}$$

where (e_1, e_2) forms an orthonormal frame of the Riemannian surface *M*. In Riemannian geometry the Gaussian curvature reflects intrinsic properties of the geodesic flow. That means properties that do not depend on the choice of local coordinates. For example the geodesics of the surface have no conjugate points if the curvature is non-positive. These geodesics are extremals of a particular time optimal control problem. Recently, Shen generalized the problem to the setting where the sea is an arbitrary Riemannian manifold $(M; h)$. When the wind is time-independent, the paths of shortest time are the geodesics of a Randers metric[9, 12]. Zermelo's navigation problem has served as a rich example of various problems in the calculus of variations for many years. It is possible to discuss a modern version of the basic problem of minimal transit time and suggest a new class of related problems[19].

Analyzing Zermelo's navigation problem and some historical problems mentioned before we can see they belong to one group of problems. Although their background is completely different. If we disregard the field of interest (discipline) they are immersed in we observe that their basic structure seems to be very similar or just the same. So the foundations of navigation can also be researched in the group of pure problems of the same origin. In this way it is possible to notice where the same root of quiet different areas began. So the evolution of one problem can be observed by analyzing the questions used to be asked in the past and their solutions or propositions. In case of navigation the process is still realized what is proved by the modern theories (including new notions and theorems). As we wrote in first sentences of the paper travelling, ordered motion started long time ago. It is difficult to state exactly when. However the point is it is continued and navigation follows the process. Its ideas and questions have changed the form many times and can be found in other disciplines as the key problems. Though it may be not easy to state they had the same origin. Of course the reader can wonder if the limits of navigation are so far. It depends on the point of view. Following above mentioned way we treat the maritime navigation as one part of the field. The historical problems (e.g. of brachystochrone) presented in the paper can be the particular cases of generalized Zermelo navigation problem where the different variables are considered, time in particular (classical case of the problem). So the new class of problems of the same origin can be researched then and analyzed for instance on different topological objects.

5 Abstraction and reference to present theories

We aim to discuss roughly a current and uniform treatment of flag and Ricci curvatures in Finsler geometry, highlighting recent developments. The flag curvature is a natural extension of the Riemannian sectional curvature to Finsler manifolds. Of particular interest are the Einstein metrics, constant Ricci curvature metrics and, as a special case, constant flag curvature metrics. Our understanding of Einstein spaces is inchoate. Much insight may be gained by considering the examples that have recently proliferated in the literature. Happily, the theory is developing as well. The Einstein and constant flag curvature metrics of spaces of Randers type, a fecund class of Finsler spaces, are now properly understood. Enlightenment comes from being able to identify the class as solutions to Zermelo's problem of navigation, a perspective that allows a very apt characterization of the Einstein spaces. When specialized to flag curvature, the navigation description yields a complete classification of the constant flag curvature Randers metrics. The sine qua non here is Shen's observation that Randers metrics may be identified with solutions to Zermelo's problem of navigation on Riemannian manifolds. This navigation structure establishes a bijection between Randers spaces (M,F) and pairs $(h;W)$ of Riemannian metrics h and vector fields W on the manifold M. The Randers metric F with navigation data $(h;W)$ is Einstein if and only if h is Einstein and W is an infinitesimal homothety of h. The transparent nature of the navigation description immediately yields a Schur lemma for the Ricci scalar, together with a certain rigidity in three dimensions. Manifold is an Abstract mathematical space which, in a close-up view, resembles the spaces described by Euclidean geometry, but which may have a more complicated structure when viewed as a whole. The surface of Earth is an example of a manifold. Locally it seems to be flat, but viewed as a whole it is round. A manifold can be constructed by gluing separate Euclidean spaces together. For example, a world map can be made by gluing many maps of local regions together, and accounting for the resulting distortions. Every space can be described by its own coordinate system, but different pieces need different ones. However, different choices of coordinate systems can be equally valid.

A sphere surface and a torus surface are examples of two-dimensional manifolds. Manifolds are important objects in mathematics and physics because they allow more complicated structures to be expressed and understood in terms of the relatively well-understood properties of simpler spaces. Additional structures are often defined on manifolds. Examples of manifolds with additional structure include:

- differentiable manifolds on which one can do calculus;
- Riemannian manifolds on which distances and angles can be defined;
- symplectic manifolds which serve as the phase space in classical mechanics;
- the four-dimensional pseudo-Riemannian manifolds which model space-time in general relativity.

The study of manifolds combines many important areas of mathematics: it generalizes concepts such as curves and surfaces as well as ideas from linear algebra and topology. Certain special classes of manifolds also have additional algebraic structure. They may behave like groups, for instance. Before the modern concept of a manifold there were several important results. Carl Friedrich Gauss may have been the first to consider Abstract spaces as mathematical objects in their own right. His theorema egregium gives a method for computing the curvature of a surface without considering the ambient space in which the surface lies. Such a surface would, in modern terminology, be called a manifold and in modern terms, the theorem proved that the curvature of the surface is an intrinsic property. Manifold theory has come to focus exclusively on these intrinsic properties (or invariants), while largely ignoring the extrinsic properties of the ambient space.

Non-Euclidean geometry considers spaces where Euclid's parallel postulate fails. Saccheri first studied them in 1733. Lobachevsky, Bolyai, and Riemann developed them 100 years later. Their research uncovered two types of spaces whose geometric structures differ from that of classical Euclidean space. These gave rise to hyperbolic geometry and elliptic geometry. In the modern theory of manifolds, these notions correspond to manifolds with constant negative and positive curvature, respectively. The spherical Earth is navigated using flat maps or charts, collected in an atlas. Similarly, a differentiable manifold can be described using mathematical maps, called coordinate charts, collected in a mathematical atlas. It is not generally possible to describe a manifold with just one chart, because the global structure of the manifold is different from the simple structure of the charts. For example, no single flat map can properly represent the entire Earth. When a manifold is constructed from multiple overlapping charts, the regions where they overlap carry information essential to understanding the global structure. In the case of a differentiable manifold, an atlas allows to do calculus on manifolds. The atlas containing all possible charts consistent with a given atlas is called the maximal atlas. Unlike an ordinary atlas, the maximal atlas of a given atlas is unique. Though it is useful for definitions, it is a very Abstract object and not used directly for example in calculations. Charts in an atlas may overlap and a single point of a manifold may be represented in several charts. If two charts overlap, parts of them represent the same region of the manifold. Given two overlapping charts, a transition function can be defined which goes from an open ball in Rn to the manifold and then back to another (or perhaps the same) open ball in Rn. The resultant map is called a change of coordinates, a coordinate transformation, a transition function, or a transition map.

If we read above mentioned notions we can see there are many ones well known in the field of navigation studied at maritime university. However it is not realized that, roughly speaking, if the programme of lectures was generalized it could treat on modern theories and important open problems in some branches of mathematics or physics. In other words the programme of navigation touches very important aspects of Abstract research. Unlike curves and surfaces, higher dimensional manifolds cannot be understood by means of visual intuition. Indeed, it is difficult or even impossible to decide whether two different descriptions of a higher-dimensional manifold refer to the same object. For this reason it is useful to develop concepts and criteria that describe intrinsic geometric and topological aspects of these mathematical objects. Such criteria are commonly referred to as being invariant, because they are the same relative to all possible descriptions of a particular manifold.

If we tried to describe the generalization of surface on which its navigational aspects are considered we ask if it is possible to generalize the manifold. It can be found out that such notions like orbifolds, algebraic varieties and schemes or CW- complexes have already been researched. An orbifold for instance is a generalization of manifold allowing for certain kinds of singularities in the topology. Roughly speaking, it is a space which locally looks like the quotients of some simple space (e.g. Euclidean space) by the actions of various finite groups. A CW complex is a topological space formed by gluing objects of different dimensionality together. For this reason they generally are not manifolds.

To measure distances and angles on manifolds, the manifold must be Riemannian. A Riemannian manifold is an analytic manifold in which each tangent space is equipped with an inner product in a manner which varies smoothly from point to point. This allows one to define various notions such as length, angles, areas (or volumes), curvature, gradients of functions and divergence of vector fields. A Finsler manifold allows the definition of distance, but not of angle. It is an analytic manifold in which each tangent space is equipped with a norm, in a manner which varies smoothly from point to point. This norm can be extended to a metric, defining the length of a curve; but it cannot in general be used to define an inner product. Any Riemannian manifold is a Finsler manifold. The Randers metrics which are considered in Finsler geometry form an important and rich class of Finsler metrics. They were first studied by physicist, G. Randers, in 1941 from the standard point of general relativity. Since then, many Finslerian geometers have made efforts in investigation on the geometric properties of Randers metrics. An important approach in discussing Randers metrics is navigation representation, that is, express Randers metric in terms of a Riemannian metric and a vector field[3]. Interestingly, the two navigation descriptions also tell us that any Einstein Randers metric that arises as a solution to Zermelo's problem of navigation on a Riemannian space form must be of constant flag curvature[9].

Zermelo’s navigation problem was to find the shortest travel time in a Riemannian manifold under the influence of a current (or wind). The solutions are the geodesics of a Randers space (a special Finsler space), and conversely, every Randers metric arises from such a problem. For example in the paper[2] the main goal is to set up a complete list up to local isometry of strongly convex Randers metrics of constant flag curvature via Zermelo navigation. We aim to present the problem here from the general point of view. That is why we try to avoid, if possible, defining strictly all used notions. Many recent developments have advanced our understanding of the flag and Ricci curvatures of Finsler metrics. Einstein metrics of Randers type are studied via their representation as solutions to Zermelo navigation on Riemannian manifolds. This viewpoint leads to the classification of all constant flag curvature Randers metrics. It also yields a Schur lemma, and settles a question of rigidity in three dimensions, for Einstein–Randers metrics. Via mentioned examples we aim to show how the problems coming from navigation are researched in the advanced and Abstract theories. There is more to be said on navigation than it is usually included in educational programmes at faculties of navigation at maritime universities. Obviously because of practical reasons there is no need to teach such advanced problems as mentioned. However it is important to know that navigation of the same origin is continuously researched although its forms can be Abstract, complicated and with surprising solutions or still open essential questions.

6 The limits of navigation conclusions

If we look back at the evolution of navigation asking the same questions but in different areas of human research we can find out that the limits have been shifted. We can consider only its maritime aspect however it is worth of remembering there are the same roots for different kinds of nowadays navigation. Because of obvious reasons we can present high technology solutions and be proud of progress made in maritime or land sectors. There are more accurate systems of better functionality. Thus, is the navigation limited by range of applications? Another question which arises here is: should we increase the area of interest in navigation at the universities? And is there any space in the programmes for navigational problems in general? Analyzing the research articles we notice that apart from very practical aspects present in the education programmes the navigational problems can be the base for modern theories with Abstract background. Are we ready to accept the conclusions implicated by them if they are even opposite to the practical solutions well known from maritime applications? After the readers' attempts to answer above mentioned questions we can ask again if there is one navigation only. And what are its limits? Do all briefly discussed Abstract problems belong to a field of navigation or they are completely isolated and became part of today mathematics and physics only? Parametric programming is one of the broadest areas of applied mathematics. Practical problems, that can be described by parametric programming, were recorded in the rock art about thirty millennia ago. As a scientific discipline, parametric programming began emerging only in the 1950's. This is done using a limited theory (mainly for linear and convex models) and by means of examples, figures, and solved real-life case studies. If we compare the evolution of parametric programming and navigation we can see that their foundations have the origin in long-ago times. However the problems of navigational aspects appeared many times through the centuries. As already said they could have different background. Interestingly, among the topics discussed in parametric programming such as games of market economy, a projectile motion model, decision making models there can also be found Zermelo's navigation problem under the water[20].

Problem of finding the shortest route has already been considered by the ancients for practical reasons. Looking back in the past it can be said that the idea came back many times but in different forms according to appearance of the new problems or circumstances. Presently the advanced studies are being carried in modern mathematical theories. Thus, the scientific research changes the researched object. In the article we tried to think of navigation in general aspect attempting to clarify the foundational aspects of the subject and thus intuitions dating back to the past centuries became precise and developed through different theories and applications. We were looking at its historical roots, the problems taken into consideration in the past which referred to navigation, however not only in its maritime sector. We compared the present education programmes at university to the most important navigational questions asked in different fields. It is worth of seeing that beside the maritime knowledge there are many areas of research interest e.g. in mathematics or physics considering exactly the same problems but with different backgrounds.

The famous mathematician Ernst Zermelo submitted in 1931 the navigational problem (Zermelo's Problem of Navigation) which is useful in modern theories for example in Finsler geometry. It was used on Riemannian manifolds to solve a long standing problem, namely the complete classification of strongly convex Randers metrics of constant flag curvature. In our opinion the navigational problems, travelling, ordered motion and related transportation go back to long-ago times. They were present with different names in the centuries following the civilization progress made in the passing time. The nature of navigation also became more Abstract. We asked if there was one navigation. Can we call the navigation the same name in practical aspects referring to practical applications and Abstract theories? Both have got the same origin. Is it still one research field or completely different disciplines? Let us think of the above mentioned problems considering their evolution through the centuries. We look at the problem of travelling with its different aspects referring to navigation. Above mentioned notions and discussed problems relating to navigation can be researched in the future applications of micro and macro scale. Such names as for instance Euler, Einstein, Riemann are not far away from the area of interest described here. The reader may well ask about the relationship between the practical and Abstract approach of navigation. The answer may be that they are the two sides of the same coin. It would be a pity if the reader of the article would conclude that there is nothing to contribute to the subject. It still has much to yield. We end this conversation by mentioning that navigation covers a wider field than usually it is thought.

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